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On free vibration of non-homogeneous transversely isotropic magneto-electro-elastic plates

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Abstract

Two independent state equations are established for transversely isotropic magneto-electro-elastic media by introducing proper stress and displacement functions. The free vibration problem of simply supported rectangular plates with general inhomogeneous (functionally graded) material properties along the thickness direction is then considered. An approximate laminate model is employed to transform the state equations with variable coefficients to the ones with constant coefficients. Two different classes of vibrations are found. In particular, the frequency of the first class is only related to the elastic property of the plate, while that of the second class is affected by the couplings among the elastic, electric and magnetic fields. Numerical results are presented and some important issues are discussed.

1. Introduction

Many researchers have paid their attention to the mechanics problems associated with magneto-electro-elastic materials recently [1–5]. As regards the structural analysis, Pan [6] derived an exact three-dimensional solution of a simply supported multilayered orthotropic magneto-electro-elastic plate using a propagator matrix method. Such a solution can play an important role in clarifying two-dimensional simplified plate theories or numerical methods. Wang et al. [7] extended the previous works on elastic and piezoelectric plates [8–10] to study the bending of multi-layered orthotropic magneto-electro-elastic rectangular plates by adopting the state space formulations. Chen and Lee [11] presented novel state space formulations for the static problem of transversely isotropic thermo-magneto-electro-elastic materials by virtue of a separation

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technique. The free vibration investigation was recently performed by Pan and Heyliger [12], who found that some natural frequencies of a multi-field (piezoelectric/magnetostrictive) plate were identical to the ones of the corresponding elastic plate. They argued that certain vibration modes of the plate were insensitive to the coupling effects among elastic, electric, and magnetic fields. The plate considered in their numerical example is in fact transversely isotropic, for which we will show theoretically in this paper that there actually exists a class of vibration, of which the frequencies depend on the elastic property only.

To this end, two independent state equations are derived from the three-dimensional dynamic equations for transversely isotropic magneto-electro-elastic materials by virtue of two separation formulations for displacements and stresses. It is noted here that this separation technique has been employed by Ding and Chen [13–15] to study the static and dynamic behaviors of piezoelastic plates and shells. Chen and Lee [11] also adopted this technique to derive two separate static state equations for transversely isotropic magento-electro-elastic materials involving thermal effect. It is seen that the state space formulations for dynamic problem are also valid when the material is inhomogeneous along the axis of symmetry. Thus the free vibration of simply supported plate that is inhomogeneous along the thickness direction is considered. To obtain the solution in an analytical form, an approximate laminate model is employed. Two independent classes of vibrations are found with the frequency of the first class being related to the elastic property of the plate only, thus validating the observation reported by Pan and Heyliger [12]. As pointed out by Li [3], the micromechanics simulation showed that the magnetoelectric coupling exists in the $BaTiO_3$–$CoFe_2O_4$ fiber reinforced or laminated plate. The effect of this coupling on the natural frequency that was not considered by Pan and Heyliger [12] is studied numerically by considering a magneto-electro-elastic plate having a functionally graded material property along the thickness direction.

2. Basic equations

Consider a transversely isotropic magneto-electro-elastic medium in Cartesian co-ordinate system (x, y, z). If z-axis is normal to the plane of isotropy, the constitutive relations are [3,4]

$$
\begin{aligned}
\sigma_x &= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \phi}{\partial z} + q_{31}\frac{\partial \psi}{\partial z},\\
\sigma_y &= c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \phi}{\partial z} + q_{31}\frac{\partial \psi}{\partial z},\\
\sigma_z &= c_{13}\frac{\partial u}{\partial x} + c_{13}\frac{\partial v}{\partial y} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \phi}{\partial z} + q_{33}\frac{\partial \psi}{\partial z},\\
\tau_{xz} &= c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \phi}{\partial x} + q_{15}\frac{\partial \psi}{\partial x},\\
\tau_{yz} &= c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + e_{15}\frac{\partial \phi}{\partial y} + q_{15}\frac{\partial \psi}{\partial y}, \quad \tau_{xy} = c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned} \tag{1}
$$

$$D_x = e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \varepsilon_{11}\frac{\partial \phi}{\partial x} - d_{11}\frac{\partial \psi}{\partial x},$$

$$D_y = e_{15}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \varepsilon_{11}\frac{\partial \phi}{\partial y} - d_{11}\frac{\partial \psi}{\partial y}, \quad (2)$$

$$D_z = e_{31}\frac{\partial u}{\partial x} + e_{31}\frac{\partial v}{\partial y} + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \phi}{\partial z} - d_{33}\frac{\partial \psi}{\partial z},$$

$$B_x = q_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - d_{11}\frac{\partial \phi}{\partial x} - \mu_{11}\frac{\partial \psi}{\partial x},$$

$$B_y = q_{15}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - d_{11}\frac{\partial \phi}{\partial y} - \mu_{11}\frac{\partial \psi}{\partial y}, \quad (3)$$

$$B_z = q_{31}\frac{\partial u}{\partial x} + q_{31}\frac{\partial v}{\partial y} + q_{33}\frac{\partial w}{\partial z} - d_{33}\frac{\partial \phi}{\partial z} - \mu_{33}\frac{\partial \psi}{\partial z},$$

where, σ_i and τ_{ij} are the normal and shear stresses, respectively; u, v and w are components of the mechanical displacement in x-, y- and z-directions, respectively; ϕ, ψ, D_i, and B_i are the electric potential, magnetic potential, electric displacement components, and magnetic induction components, respectively; c_{ij}, ε_{ij}, e_{ij}, q_{ij}, d_{ij}, and μ_{ij} are the elastic, dielectric, piezoelectric, piezomagnetic, magnetoelectric, and magnetic constants, respectively. For transversely isotropic material, the relation $c_{11} = c_{12} + 2c_{66}$ holds. In this paper, all these material constants are assumed to be functions of the co-ordinate variable z. The equations of motion are [12]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho\frac{\partial^2 v}{\partial t^2}, \quad (4)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2},$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0, \quad (5)$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \quad (6)$$

where ρ is the density of the material, also a function of z.

Following a routine derivation [7,16], we can derive the conventional state equation involving inertia effect as well as the material inhomogeneity along z-direction from Eqs. (1)–(6) as follows:

$$\frac{\partial}{\partial z}[u, v, D_z, B_z, \sigma_z, \tau_{xz}, \tau_{yz}, \phi, \psi, w]^{\mathrm{T}} = \begin{bmatrix} \mathbf{0} & \mathbf{A}_1 \\ \mathbf{A}_2 & \mathbf{0} \end{bmatrix} [u, v, D_z, B_z, \sigma_z, \tau_{xz}, \tau_{yz}, \phi, \psi, w]^{\mathrm{T}}, \tag{7}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \frac{1}{c_{44}} & 0 & -\frac{e_{15}}{c_{44}}\frac{\partial}{\partial x} & -\frac{q_{15}}{c_{44}}\frac{\partial}{\partial x} & -\frac{\partial}{\partial x} \\ & \frac{1}{c_{44}} & -\frac{e_{15}}{c_{44}}\frac{\partial}{\partial y} & -\frac{q_{15}}{c_{44}}\frac{\partial}{\partial y} & -\frac{\partial}{\partial y} \\ & & k_1 \Lambda & k_2 \Lambda & 0 \\ & \text{sym.} & & k_3 \Lambda & 0 \\ & & & & \rho \frac{\partial^2}{\partial t^2} \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} \rho\frac{\partial^2}{\partial t^2} + k_4\frac{\partial^2}{\partial x^2} - c_{66}\frac{\partial^2}{\partial y^2} & (k_4 + c_{66})\frac{\partial^2}{\partial x \partial y} & -g_2\frac{\partial}{\partial x} & -g_3\frac{\partial}{\partial x} & -g_1\frac{\partial}{\partial x} \\ & \rho\frac{\partial^2}{\partial t^2} + k_4\frac{\partial^2}{\partial y^2} - c_{66}\frac{\partial^2}{\partial x^2} & -g_2\frac{\partial}{\partial y} & -g_3\frac{\partial}{\partial y} & -g_1\frac{\partial}{\partial y} \\ & & \frac{\alpha_{22}}{\alpha} & \frac{\alpha_{23}}{\alpha} & \frac{\alpha_{12}}{\alpha} \\ \text{sym.} & & & \frac{\alpha_{33}}{\alpha} & \frac{\alpha_{13}}{\alpha} \\ & & & & \frac{\alpha_{11}}{\alpha} \end{bmatrix} \tag{8}$$

in which $\Lambda = \partial^2/\partial x^2 + \partial^2/\partial y^2$, “sym.” indicates a symmetric matrix,

$$k_1 = \varepsilon_{11} + e_{15}^2/c_{44}, \quad k_2 = d_{11} + e_{15}q_{15}/c_{44}, \quad k_3 = \mu_{11} + q_{15}^2/c_{44},$$

$$k_4 = c_{13}g_1 + e_{31}g_2 + q_{31}g_3 - c_{11}, \quad g_i = (c_{13}\alpha_{1i} + e_{31}\alpha_{2i} + q_{31}\alpha_{3i})/\alpha,$$

$$\alpha = \begin{vmatrix} c_{33} & e_{33} & q_{33} \\ e_{33} & -\varepsilon_{33} & -d_{33} \\ q_{33} & -d_{33} & -\mu_{33} \end{vmatrix} \tag{9}$$

and α_{ij} are the corresponding algebraic cofactors of α with $\alpha_{ij} = \alpha_{ji}$.

3. Alternative state space formulations

To construct new state space formulations, the following substitutions are employed [11,13–15]:

$$u = -\frac{\partial \Psi}{\partial y} - \frac{\partial G}{\partial x}, \quad v = \frac{\partial \Psi}{\partial x} - \frac{\partial G}{\partial y}, \quad \tau_{xz} = -\frac{\partial \tau_1}{\partial y} - \frac{\partial \tau_2}{\partial x}, \quad \tau_{yz} = \frac{\partial \tau_1}{\partial x} - \frac{\partial \tau_2}{\partial y}, \tag{10}$$

where Ψ and G are two displacement functions, and τ_1 and τ_2 are two stress functions. By virtue of Eq. (10), similar to Ref. [11], we can arrive at the following state equations:

$$\frac{\partial}{\partial z}\begin{Bmatrix} \Psi \\ \tau_1 \end{Bmatrix} = \begin{bmatrix} 0 & 1/c_{44} \\ \rho \partial^2/\partial t^2 - c_{66}\Lambda & 0 \end{bmatrix} \begin{Bmatrix} \Psi \\ \tau_1 \end{Bmatrix}, \tag{11}$$

$$\frac{\partial}{\partial z}\begin{Bmatrix} G \\ \sigma_z \\ D_z \\ B_z \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{c_{44}} & 1 & \dfrac{e_{15}}{c_{44}} & \dfrac{q_{15}}{c_{44}} \\ \Lambda & \rho \dfrac{\partial^2}{\partial t^2} & 0 & 0 \\ \dfrac{e_{15}}{c_{44}}\Lambda & 0 & k_1\Lambda & k_2\Lambda \\ \dfrac{q_{15}}{c_{44}}\Lambda & 0 & k_2\Lambda & k_3\Lambda \end{bmatrix} \begin{Bmatrix} \tau_2 \\ w \\ \phi \\ \psi \end{Bmatrix}, \tag{12}$$

$$\frac{\partial}{\partial z}\begin{Bmatrix} \tau_2 \\ w \\ \phi \\ \psi \end{Bmatrix} = \begin{bmatrix} \rho \dfrac{\partial^2}{\partial t^2} + k_4\Lambda & g_1 & g_2 & g_3 \\ g_1\Lambda & \beta_{11} & \beta_{12} & \beta_{13} \\ g_2\Lambda & \beta_{12} & \beta_{22} & \beta_{23} \\ g_3\Lambda & \beta_{13} & \beta_{23} & \beta_{33} \end{bmatrix} \begin{Bmatrix} G \\ \sigma_z \\ D_z \\ B_z \end{Bmatrix}, \tag{13}$$

where $\beta_{ij} = \alpha_{ij}/\alpha$. The remainder variables are determined by

$$\sigma_x + \sigma_y = 2(k_4 + c_{66})\Lambda G + 2g_1\sigma_z + 2g_2 D_z + 2g_3 B_z,$$

$$\sigma_x - \sigma_y = -2c_{66}\left[2\frac{\partial^2 \Psi}{\partial x \partial y} + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)G\right], \quad \tau_{xy} = c_{66}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\Psi - 2\frac{\partial^2 G}{\partial x \partial y}\right],$$

$$D_x = -\frac{e_{15}}{c_{44}}\frac{\partial \tau_1}{\partial y} - \frac{\partial}{\partial x}\left(k_1\phi + k_2\psi + \frac{e_{15}}{c_{44}}\tau_2\right),$$

$$D_y = \frac{e_{15}}{c_{44}}\frac{\partial \tau_1}{\partial x} - \frac{\partial}{\partial y}\left(k_1\phi + k_2\psi + \frac{e_{15}}{c_{44}}\tau_2\right), \tag{14}$$

$$B_x = -\frac{q_{15}}{c_{44}}\frac{\partial \tau_1}{\partial y} - \frac{\partial}{\partial x}\left(k_2\phi + k_3\psi + \frac{q_{15}}{c_{44}}\tau_2\right),$$

$$B_y = \frac{q_{15}}{c_{44}}\frac{\partial \tau_1}{\partial x} - \frac{\partial}{\partial y}\left(k_2 v + k_3\psi + \frac{q_{15}}{c_{44}}\tau_2\right).$$

It can be seen that the ten state variables Ψ, τ_1, G, σ_z, D_z, B_z, τ_2, w, ϕ and ψ are sorted into two groups: One group is related to Ψ and τ_1 only, and the other to the remaining eight state variables. It will be demonstrated later in this paper that the separation of state equations can help us to understand some particular characteristics occupied by practical problems that cannot be revealed by the tenth order state equation, i.e., Eq. (7).

4. Vibration analysis of non-homogeneous rectangular plates

Consider a simply supported, transversely isotropic, rectangular plate of width a, length b and thickness H (Fig. 1a), with its isotropic plane parallel to the middle plane. The plate is assumed to be inhomogeneous along the thickness direction.

The simply supported boundary conditions for a magneto-electro-elastic plate can be expressed as follows:

$$x = 0, a\colon w = \sigma_x = v = \phi = \psi = 0 \quad \text{and} \quad y = 0, b\colon w = \sigma_y = u = \phi = \psi = 0. \tag{15}$$

To satisfy the above conditions, we can assume

$$\begin{Bmatrix} \Psi \\ \tau_1 \end{Bmatrix} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \begin{Bmatrix} H^2 \bar{\Psi}(\zeta) \\ H c_{44}^0 \bar{\tau}_1(\zeta) \end{Bmatrix} \cos(m\pi\xi)\cos(n\pi\eta)\exp(\mathrm{i}\omega t), \tag{16}$$

$$\begin{Bmatrix} G \\ \sigma_z \\ D_z \\ B_z \\ \tau_2 \\ w \\ \phi \\ \psi \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} H^2 \bar{G}(\zeta)/K \\ c_{44}^0 \bar{\sigma}_z(\zeta) \\ \sqrt{c_{44}^0 \varepsilon_{33}^0}\bar{D}_z(\zeta) \\ \sqrt{c_{44}^0 \mu_{33}^0}\bar{B}_z(\zeta) \\ H c_{44}^0 \bar{\tau}_2(\zeta) \\ H\bar{w}(\zeta) \\ H\sqrt{c_{44}^0/\varepsilon_{33}^0}\bar{\phi}(\zeta) \\ H\sqrt{c_{44}^0/\mu_{33}^0}\bar{\psi}(\zeta) \end{Bmatrix} \sin(m\pi\xi)\sin(n\pi\eta)\exp(\mathrm{i}\omega t), \tag{17}$$

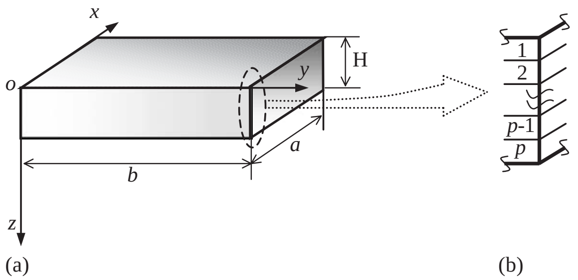

Fig. 1. A non-homogeneous rectangular plate and the approximate laminate model.

where $\zeta = z/H$, $\xi = x/a$ and $\eta = y/b$ are the dimensionless co-ordinates, $K = -(s_1^2 + s_2^2)$, $s_1 = (H/a)m\pi$, $s_2 = (H/b)n\pi$, ω is the circular frequency, and c_{44}^0, e_{33}^0 and ε_{33}^0, etc. represent the material constants at $z = 0$ (the top surface).

Substitution of Eqs. (16) and (17) into Eqs. (11)–(13), gives for an arbitrary couple of (m, n)

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{V}_1(\zeta) = \mathbf{M}_1\mathbf{V}_1(\zeta), \quad \frac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{V}_2(\zeta) = \mathbf{M}_2\mathbf{V}_2(\zeta), \tag{18}$$

where $\mathbf{V}_1 = [\bar{\Psi}, \bar{\tau}_1]^{\mathrm{T}}$, $\mathbf{V}_2 = [\bar{G}, \bar{\sigma}_z, \bar{D}_z, \bar{B}_z, \bar{\tau}_2, \bar{w}, \bar{\phi}, \bar{\psi}]^{\mathrm{T}}$, and

$$\mathbf{M}_1 = \begin{bmatrix} 0 & \dfrac{c_{44}^0}{c_{44}} \\ -\dfrac{\rho}{\rho^0}\Omega^2 - \dfrac{c_{66}K}{c_{44}^0} & 0 \end{bmatrix},$$

$$\mathbf{M}_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & \dfrac{c_{44}^0}{c_{44}}K & K & \dfrac{e_{15}K}{c_{44}}\sqrt{\dfrac{c_{44}^0}{\varepsilon_{33}^0}} & \dfrac{q_{15}K}{c_{44}}\sqrt{\dfrac{c_{44}^0}{\mu_{33}^0}} \\
0 & 0 & 0 & 0 & & -\dfrac{\rho}{\rho^0}\Omega^2 & 0 & 0 \\
0 & 0 & 0 & 0 & & & \dfrac{k_1}{\varepsilon_{33}^0}K & \dfrac{k_2K}{\sqrt{\mu_{33}^0\varepsilon_{33}^0}} \\
0 & 0 & 0 & 0 & \text{sym.} & & & \dfrac{k_3}{\mu_{33}^0}K \\
-\dfrac{\rho}{\rho^0}\dfrac{\Omega^2}{K} + \dfrac{k_4}{c_{44}^0} & g_1 & g_2\sqrt{\dfrac{\varepsilon_{33}^0}{c_{44}^0}} & g_3\sqrt{\dfrac{\mu_{33}^0}{c_{44}^0}} & 0 & 0 & 0 & 0 \\
 & \beta_{11}c_{44}^0 & \beta_{12}\sqrt{c_{44}^0\varepsilon_{33}^0} & \beta_{13}\sqrt{c_{44}^0\mu_{33}^0} & 0 & 0 & 0 & 0 \\
 & & \beta_{22}\varepsilon_{33}^0 & \beta_{23}\sqrt{\varepsilon_{33}^0\mu_{33}^0} & 0 & 0 & 0 & 0 \\
\text{sym.} & & & \beta_{33}\mu_{33}^0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{19}$$

where $\Omega = \omega H\sqrt{\rho^0/c_{44}^0}$ is the dimensionless frequency and ρ^0 is the density at $z = 0$. Since the material properties vary with the variable z, it is usually difficult to obtain the solutions to the two state equations in Eq. (18) directly except for some particular cases. For the general case that the material constants are arbitrary functions of z (can be different for different material constants), an efficient analysis based on the approximate laminate model, as shown in Fig. 1b, can be adopted [17,18,11]. In this method, the plate is divided into p equal thin layers (Fig. 1b), each with a small thickness. In every layer, the material constants can be assumed constant. Consequently, the coefficient matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are constant and will be denoted as $\mathbf{M}_1^j$ and $\mathbf{M}_2^j$ in the jth layer, which have the values at the middle plane of that layer. It is clear that with the number of layers increasing, the laminate model gradually approaches the actual plate and the solution becomes more and more close to the exact one. By virtue of the matrix theory, we can obtain the solutions of the two state equations within each layer, from which the following relations can be finally established:

$$\mathbf{V}_1(1) = \mathbf{T}_1\mathbf{V}_1(0), \quad \mathbf{V}_2(1) = \mathbf{T}_2\mathbf{V}_2(0), \tag{20}$$

where $\mathbf{T}_1 = \prod_{j=p}^{1} \exp(\mathbf{M}_1^j/p)$ and $\mathbf{T}_2 = \prod_{j=p}^{1} \exp(\mathbf{M}_2^j/p)$ are matrices of the second and eighth order, respectively. The derivation of Eq. (20) is very straightforward and the readers are referred to Refs. [11,16] for details.

For the free vibration problem, apart from the vanishing mechanical forces and magnetic induction ($\bar{\sigma}_z = \bar{\tau}_1 = \bar{\tau}_2 = \bar{B}_z = 0$) at the top and bottom surfaces, we consider two different types of electric conditions, i.e., the closed-circuit ($\bar{\phi} = 0$) and open-circuit ($\bar{D}_z = 0$) conditions. For the closed-circuit condition, we can obtain from Eq. (20) the following two frequency equations on the requirement of existing non-trivial solutions:

$$T_{121} = 0, \tag{21}$$

$$\begin{vmatrix} T_{221} & T_{223} & T_{226} & T_{228} \\ T_{241} & T_{243} & T_{246} & T_{248} \\ T_{251} & T_{253} & T_{256} & T_{258} \\ T_{271} & T_{273} & T_{276} & T_{278} \end{vmatrix} = 0, \tag{22}$$

where T_{kij} are elements of the matrix $\mathbf{T}_k$. Now we have obtained two independent frequency equations: Eq. (21), only involving the elastic property of the plate, corresponds to a purely in-plane vibration, while Eq. (22), depending on all material properties, corresponds to a general flexural vibration. For the open-circuit condition, we find that the frequency equation for the in-plane vibration is the same as Eq. (21), while that for the general flexural vibration becomes

$$\begin{vmatrix} T_{221} & T_{226} & T_{227} & T_{228} \\ T_{231} & T_{236} & T_{237} & T_{238} \\ T_{241} & T_{246} & T_{247} & T_{248} \\ T_{251} & T_{256} & T_{257} & T_{258} \end{vmatrix} = 0. \tag{23}$$

The fact that there exist two independent classes of vibrations mentioned above strictly affirms the observation obtained by Pan and Heyliger [12].

As pointed out by Pan and Heyliger [12], other possible conditions such as mechanically fixed and/or zero magnetic potential at the two surfaces also can be treated. These are omitted in this paper for simplicity.

For the calculation of vibration modes, the following equation should be employed to compute the state variables at an arbitrary ζ:

$$\mathbf{V}_k(\zeta) = \exp[\mathbf{M}_k^j(\zeta - \zeta_j)] \prod_{i=j-1}^{1} \exp[\mathbf{M}_k^i/p]\mathbf{V}_k(0) \quad (k = 1, 2; \zeta_j \leqslant \zeta \leqslant \zeta_{j+1}), \tag{24}$$

where $\zeta_j = (j-1)/p$. The remainder variables are then determined from Eq. (14).

5. Numerical results

In all numerical examples to be considered, we take $m = n = 1$, which represents the fundamental vibrational mode that is of practical importance [19]. The example of sandwich piezoelectric and magnetostrictive plate (open-circuit at the top and bottom surfaces) of $H/a = H/b = 1$ studied by Pan and Heyliger [12] is considered here for comparison. The material constants of $BaTiO_3$ and $CoFe_2O_4$ can be found in Tables 4 and 5 in Pan and Heyliger [12] (the densities of the two materials are assumed identical). Table 1 gives the lowest five frequency parameters $\omega^* = \omega a\sqrt{\rho_{\max}/C_{\max}}$ ($C_{\max}$ being the maximum of the c_{ij} in the whole sandwich plate and $\rho_{\max} = 1$), which was defined by Pan and Heyliger [12], for the first and second classes of vibrations. It can be seen that the frequency of Mode 1 given by Pan and Heyliger [12] is actually the lowest frequency of the first class of vibration. This can also be verified by Fig. 1 in Pan and Heyliger [12], where the transverse displacement, electric potential and magnetic potential all vanish, for Mode 1 of the B only plate. Note that our results of the lowest frequency of the first class of vibration are identical with Pan and Heyliger's calculations for B only and F only plates and the difference of results for B/F/B/ and F/B/F plates is also negligible. However, the discrepancy, generally less than 3%, can be observed for other higher modes. To check our results, we also have performed the calculations directly based on Eq. (7), i.e., using the conventional state space formulations, whose accuracy, correctness and effectiveness have been verified by various researchers for elastic, piezoelectric, and magneto-electro-elastic plates and shells [7–10,16]. We actually have obtained exactly the same results as that presented in Table 1. It is pointed out here that by setting $q_{ij} = 0$ and/or $e_{ij} = 0$ in relevant formulations, the present method can be directly

Table 1
Lowest 5 frequency parameters $\omega^* = \omega a\sqrt{\rho_{max}/C_{max}}$ of the sandwich plate studied in Pan and Heyliger [12][a]

Order	B only		F only		B/F/B		F/B/F	
	1st	2nd	1st	2nd	1st	2nd	1st	2nd
1	2.30033	2.10902 (2.08137)	1.97472	1.54009	1.82648	1.54742 (1.53366)	1.89865	1.60543 (1.60524)
2	2.80145	2.81507 (2.74968)	2.33726	2.25432	2.15561	2.24483 (2.23332)	2.31557	2.24751 (2.24737)
3	3.93927	3.96097 (3.83200)	3.18631	3.21210	3.07652	3.08342 (3.02221)	3.11555	3.22160 (3.22148)
4	5.31985	4.38852 (4.30270)	4.23897	3.78466	4.11470	3.44376 (3.34520)	4.17674	3.73693 (3.73691)
5	6.79683	5.50595 (5.40683)	5.37695	4.52726	5.24651	4.39150 (4.25627)	5.30704	4.39757 (4.39755)

[a] Open-circuit electric condition was adopted in Ref. [12]; the results for the closed-circuit condition are also given here in parentheses for comparison.

used to calculate the natural frequencies of piezoelastic, magnetoelastic as well as purely elastic plates. In fact, the results for a piezoelectric PZT-4 plate and the corresponding elastic plate (see Table 3 in Ref. [12]) have been successfully reproduced.

From Table 1, we also find that the lowest natural frequency of the plate usually belongs to the second class of vibration. This property also has been observed for the piezoelectric plate [13]. Table 1 also lists the natural frequencies for the closed-circuit electric condition for the purpose of comparison. Note that since the F only plate has $e_{ij} = 0$, the surface electric condition has no effect on its frequencies. It is seen that the frequencies for the closed-circuit condition are usually lower than the corresponding ones for the open-circuit condition. This phenomenon has been well reported in the dynamic analyses of piezoelectric plates and shells [15].

Now we consider a simply supported non-homogeneous magneto-electro-elastic rectangular plate with $H/a = 1/4$ and $H/b = 1/6$. The following functionally graded model [20,21] regarding the material inhomogeneity is employed:

$$m_{ij} = m_{ij}^B \left(\frac{H-z}{H} \right)^{\kappa} + m_{ij}^C \left[1 - \left(\frac{H-z}{H} \right)^{\kappa} \right], \tag{25}$$

where κ is the gradient index, and m_{ij}^B and m_{ij}^C are the material constants of $BaTiO_3$ and $CoFe_2O_4$, respectively. It is noted that when $\kappa = 0$, the plate is a homogeneous $BaTiO_3$ plate and when κ tends to infinite, it becomes a homogeneous $CoFe_2O_4$ plate. At the first stage, we take the material constants as exactly the same as those in Pan and Heyliger [12], where the magnetoelectric coupling was not considered, i.e., $d_{11} = d_{33} = 0$. It is also mentioned that we just consider a special kind of material inhomogeneity from the theoretical point of view, regardless of its background in engineering. However, one can expect that magneto-electro-elastic materials with functionally graded property will appear soon just as other advanced materials [22].

Table 2 compares the calculated lowest five dimensionless frequencies $\Omega = \omega H \sqrt{\rho^0 / c_{44}^0}$ of the FGM plate for a 29- and a 30-layer model, respectively. The gradient index is taken to be $\kappa = 2$. It is seen that the difference between the results of two models is completely negligible. Thus in the following, we shall take $p = 30$ and the results are believed to be of high accuracy. From Table 2,

Table 2
Lowest 5 dimensionless frequencies Ω of a functionally graded magneto-electro-elastic plate for two laminate models ($\kappa = 2.0$)

Electric condition	p	Class	Frequency order				
			1	2	3	4	5
Open	29	1st	1.04277	3.36482	6.47999	9.64833	12.8308
		2nd	0.431788	1.83818	3.79365	6.35670	8.02087
	30	1st	1.04277	3.36482	6.47999	9.64833	12.8308
		2nd	0.431785	1.83818	3.79364	6.35670	8.02088
Closed	29	2nd	0.430725	1.83224	3.71108	6.27877	7.95984
	30	2nd	0.430723	1.83224	3.71108	6.27877	7.95985

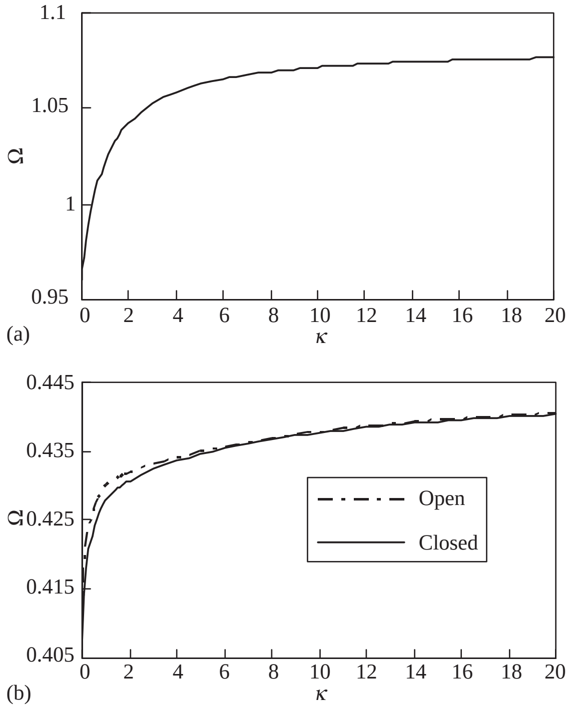

Fig. 2. Variations of the lowest non-dimensional frequency Ω versus the gradient index κ: (a) the first class and (b) the second class.

it is also seen that the electric condition at the plate surfaces has a little effect on the natural frequencies.

The variations of the lowest dimensionless frequency Ω with the gradient index κ are shown in Fig. 2. Since the first class of vibration depends on the elastic property of the plate only, the electric condition on the plate surfaces does not affect the associated frequencies, and hence only one curve is given in Fig. 2(a). As we can see, Ω increases with κ. In fact, it changes gradually from the one of the homogeneous $BaTiO_3$ plate to that of the homogeneous $CoFe_2O_4$ plate. From Fig. 2(b), we find that the difference between frequencies for the two electric conditions also varies with κ. In particular, for the homogeneous $BaTiO_3$ plate, the difference is most significant, while it decreases with increasing κ.

Li [3] predicted from a micromechanics analysis that for a two-phase $BaTiO_3$–$CoFe_2O_4$ composite, the magnetoelectric coefficients are not zero ($d_{11} \neq 0, d_{33} \neq 0$), although neither phase shows this coupling. The magnitude of the magnetoelectric coupling depends on the factors such as the material combination method and phase volume fraction. Our recent investigation showed that, when the plate is subjected to static magnetic load, the effect of magnetoelectric coupling should be taken into consideration [11]. Now we try to study the effect of this coupling on the natural frequencies of the inhomogeneous magneto-electro-elastic plate. Note that the material constants of the functionally graded plate represented by Eq. (25) vary from the ones of $BaTiO_3$ at the top surface to that of $CoFe_2O_4$ at the bottom surface continuously. Thus the two coefficients

Table 3
Effect of magnetoelectric coupling on natural frequencies Ω of the second class of vibration of the functionally graded plate ($\kappa = 0.5$)

Electric condition	Coupling	Frequency order				
		1	2	3	4	5
Open	No	0.426666	1.77514	3.87899	6.11117	7.54853
	Yes	0.426669	1.77516	3.87900	6.11112	7.54816
Closed	No	0.424197	1.72746	3.65445	6.07629	7.35026
	Yes	0.424201	1.72746	3.65445	6.07626	7.35012

d_{11} and d_{33} are assumed to vary with the thickness direction in a way presented in Fig. 5(a) of Li [3], which is fitted here as [11]

$$d_{11}(\zeta) = 3.5\zeta_c^3 - 35\zeta_c^2 + 31.5\zeta_c,$$
$$d_{33}(\zeta) = \begin{cases} -2250\zeta_c^2 + 5025\zeta_c & \text{for } 0 \leqslant \zeta \leqslant 0.9, \\ -300\,000\zeta_c^2 + 543\,000\zeta_c - 243\,000 & \text{for } 0.9 \leqslant \zeta \leqslant 1.0 \end{cases} \tag{26}$$

with the unit 10^{-12} Ns/(VC), where $\zeta_c = 1 - \zeta$. In addition, $\kappa = 0.5$ is selected for performing the numerical calculation. Table 3 compares the lowest five non-dimensional frequencies between two different cases (with and without the magnetoelectric coupling). It is shown that for both types of electric conditions, the magnetoelectric coupling almost has no influence on the natural frequencies of the plate. In fact, we have calculated the lowest 40 natural frequencies, and the same conclusion is obtained. Thus, for most applications, the magnetoelectric coupling could be neglected in the frequency analysis of $BaTiO_3$–$CoFe_2O_4$ composite structures.

The final example considers the free vibration of the above functionally graded plate with $a = b$ for different values of H/a. The open-circuit electric condition is assumed at the top and bottom surfaces. The results are given in Table 4 for both classes of vibration. Note that the effect of magnetoelectric coupling is found very small and indeed can be neglected, just as shown in the last example. Thus only results with magnetoelectric coupling are given in Table 4. If the thickness of the plate is fixed, the definition of Ω keeps unchanged. Then, it is seen from Table 4 that, with the increase of the in-plane size a, the lowest frequency of the first class and the lowest two ones of the second class decrease rapidly. However, the other high order frequencies are not so sensitive to the in-plane size as the three lowest frequencies mentioned above.

6. Concluding remarks

Two separated state equations derived from the three-dimensional transversely isotropic magneto-electro-elasticity are employed to study the free vibration of a simply supported magneto-electro-elastic plate that is inhomogeneous along the thickness direction. For the sake of

Table 4
Natural frequencies Ω of a square functionally graded plate (open-circuit) for different thickness-to-span ratios ($\kappa = 0.5$)

H/a	Class	Frequency order				
		1	2	3	4	5
0.3	1	1.41503	3.47095	6.49336	9.61131	12.7547
	2	0.768807	2.47386	4.27749	6.03656	7.91301
0.2	1	0.943521	3.30556	6.40687	9.55316	12.7110
	2	0.383198	1.67340	3.82879	6.12678	7.50010
0.1	1	0.471810	3.20227	6.35442	9.51811	12.6846
	2	0.104730	0.843201	3.50988	6.27603	7.16449
0.05	1	0.235911	3.17593	6.34124	9.50932	12.6781
	2	0.0268721	0.422374	3.42161	6.34323	7.05138
0.02	1	0.0943653	3.16851	6.33754	9.50686	12.6762
	2	0.0043324	0.169035	3.39613	6.36691	7.01489

convenience, the approximate laminate model is adopted in the analysis, which allows us to deal with an arbitrary material inhomogeneity.

Two independent classes of vibrations are then found. The first class is purely in-plane that is characterized by the vanishing of transverse displacement, electric potential and magnetic potential; the corresponding bulk strain also equals zero. The second class of vibration is generally flexural, having both in-plane and out-of-plane displacements as well as non-zero electric and magnetic potentials. Furthermore, the frequency of the first class is independent of the magnetic and electric parameters of the plate, thus giving a strict verification of the observation reported by Pan and Heyliger [12]. It is also emphasized here that the electric displacements and magnetic inductions do not vanish in the plate even for the first class of vibration because of the piezoelectric and piezomagnetic couplings, as shown in Eqs. (2) and (3), respectively. Note that by setting $q_{ij} = 0$ and/or $e_{ij} = 0$, the formulations presented in this paper can be directly employed to analyze the free vibration of magnetoelastic, piezoelastic as well as purely elastic plates.

Numerical investigation shows that the method converges rapidly and is very accurate. In fact, the solution presented in the paper based on the approximate laminate model will gradually approach the exact solution of the original inhomogeneous plate when the number of layers increases. Thus, it can serve as a three-dimensional benchmark solution to check various two-dimensional approximate theories and numerical methods. It is known that there is magneto-electric coupling between the two individual phases in the $BaTiO_3$–$CoFe_2O_4$ composite [3]. Our numerical result shows that this kind of coupling almost has no effect on the lower order natural frequencies of the plate under consideration.

It should be noted that for a homogeneous or laminated plate, the solution obtained in this paper becomes completely exact as shown by Wang et al. [7] for the bending problem. Furthermore, it is pointed out here that exact solution can also be obtained when the material inhomogeneity obeys the following special rule:

$$m_{ij}(\zeta) = m_{ij}^0 \exp(\gamma\zeta), \tag{27}$$

where m_{ij} can be an arbitrary material constant including the density, m_{ij}^0 and γ are known constants. We should assume the following form of solution instead of Eqs. (16) and (17):

$$\begin{Bmatrix} \Psi \\ \tau_1 \end{Bmatrix} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \begin{Bmatrix} H^2 \bar{\Psi}(\zeta) \\ H c_{44} \bar{\tau}_1(\zeta) \end{Bmatrix} \cos(m\pi\xi)\cos(n\pi\eta)\exp(\mathrm{i}\omega t), \tag{28}$$

$$\begin{Bmatrix} G \\ \sigma_z \\ D_z \\ B_z \\ \tau_2 \\ w \\ \phi \\ \psi \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} H^2 \bar{G}(\zeta)/J_{mn} \\ c_{44}\bar{\sigma}_z(\zeta) \\ \sqrt{c_{44}\varepsilon_{33}}\bar{D}_z(\zeta) \\ \sqrt{c_{44}\mu_{33}}\bar{B}_z(\zeta) \\ H c_{44}\bar{\tau}_2(\zeta) \\ H\bar{w}(\zeta) \\ H\sqrt{c_{44}/\varepsilon_{33}}\bar{\phi}(\zeta) \\ H\sqrt{c_{44}/\mu_{33}}\bar{\psi}(\zeta) \end{Bmatrix} \sin(m\pi\xi)\sin(n\pi\eta)\exp(\mathrm{i}\omega t). \tag{29}$$

Then we can find that state equations with constant coefficients can be derived, and exact solutions become obtainable.

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